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Separability Structures and Killing–Yano Tensors in Vacuum Type-*D* Space-Times without Acceleration[1]

M. Demianski

Institute of Theoretical Physics, University of Warsaw, Poland

M. Francaviglia

Institute of Mathematical Physics, University of Torino, Italy

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Relationships among the existence of Killing tensors, Killing–Yano tensors, and separability structures with two Killing vectors in vacuum type-*D* space-times are investigated. It is proved that the existence of those objects is equivalent with the assumption that space-time is without acceleration.

1. INTRODUCTION

In recent years the theory of separability of the Hamilton-Jacobi equation for geodesics has become interesting in general relativity, especially for space-times possessing suitable algebraic properties or suitable symmetries. A first important class of separable space-times, including several type-*D* solutions, was discovered in 1968 by B. Carter (1969). From the beginning, many people realized that in separability theory geometrical objects called *Killing tensors* (Hugston and Sommers, 1973; Walker and Penrose, 1970; Woodhouse, 1975) play an important role. The role of Killing tensors was further clarified by Benenti with the introduction of so-called *separability structures* of type $\mathfrak{S}_r$ (Benenti, 1975/1976; Benenti and Francaviglia, 1980). Other objects whose existence is related to the existence of Killing tensors and to the symmetries of some known separable space-time are so-called *Killing–Yano tensors*, which have recently been investigated by Collinson (1976) and Collinson and Smith (1977) and by

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Stephani (1978). Finally, it should be also mentioned that the Killing tensors appearing in all of Carter's separable space-times have *Segrè characteristic* [(11)(11)].

The literature on these problems is rather dispersed. In the recent review paper (Benenti and Francaviglia, 1980) a first unified treatment was presented. The aim of this paper is to point out that for vacuum type-D space-times the existence of a Killing tensor, the existence of a Killing–Yano tensor, and the existence of a regular $\mathbb{S}_2$-separability structure of Carter's type are equivalent to the property of (V_4, g) being without acceleration.[2] Instead of giving a formally organized proof we shall deduce such equivalence from several remarks concerning already known facts.

2. VACUUM TYPE-D SPACE-TIMES AND SEPARABILITY

In a recent paper we investigated the existence of Killing tensors[3] in vacuum type-D space-times and we proved that all such solutions without acceleration admit a K tensor (Demianski and Francaviglia, 1980). We also proved that whenever this K tensor exists it fits into a separability structure for (V_4, g). We recall that a separability structure of type $\mathbb{S}_r$ in a pseudo-Riemannian manifold (V_n, g), with $n \geqslant r$, is an equivalence class of coordinate charts in which the Hamilton–Jacobi equation for geodesics is separable with (at most) r ignorable coordinates. A theorem due to Benenti (1975/1976, 1980) and Benenti and Francaviglia (1980) characterizes (regular) $\mathbb{S}_r$ structures as follows[4]:

Theorem. A manifold (V_n, g) admits a (regular) $\mathbb{S}_r$ structure if and only if it admits r commuting K vectors $\underset{\alpha}{X}$ $(\alpha = n-r+1, \ldots, n)$ and $n-r$ K tensors $\underset{a}{K}$ $(a = 1, \ldots, n-r)$, all of them independent, which satisfy the following conditions:

(i) in the Lie algebra of K tensors with Schouten–Nijenhuis brackets[5] the commutation relations

$$\left[\underset{a}{K}, \underset{b}{K}\right] = 0 \tag{2.1}$$

[2]Vacuum type-D space-times may be conveniently classified by four independent real parameters, viz., mass, rotation, acceleration, and Newman–Unti–Tamburino parameter (see, e.g., Demianski and Plebanski, 1976).

[3]Hereafter abbreviated K tensor.

[4]For the definition of regular separability structure see Benenti and Francaviglia (1980).

[5]The Schouten–Nijenhuis brackets are defined by

$$\tfrac{1}{2}[H, K]^{ijl} = H^{m(i}\nabla_m K^{jl)} - K^{m(i}\nabla_m H^{jl)}$$

$$\left[\underset{a}{K},\underset{\alpha}{X}\right]=0, \qquad \forall a,b,\alpha \tag{2.2}$$

hold.

(ii) the K tensors $\underset{a}{K}$ have in common $n-r$ eigenvectors $\underset{a}{X}$ such that

$$\left[\underset{a}{X},\underset{b}{X}\right]=\left[\underset{a}{X},\underset{\alpha}{X}\right]=0, \qquad \forall a,b,\alpha \tag{2.3}$$

$$g\left(\underset{a}{X},\underset{\alpha}{X}\right)=0, \qquad \forall a,\alpha \tag{2.4}$$

In the sequel we shall systematically omit the adjective "regular" and those structures will be simply called separability structures. We remark that from the theorem it follows that the metric tensor g always appears among the K tensors $\underset{a}{K}$.

Here we are interested in the case $n=4$, $r=2$, with g of Lorentzian signature, i.e., in the case of $\mathbb{S}_2$ separability structures in space-time (V_4, g).

In Benenti and Francaviglia (1979) it was shown that the metric of a $\mathbb{S}_2$-separable space-time (V_4, g) can be always reduced (in so-called normal coordinates) to its *canonical form*:

$$g^{aa}=\frac{\psi_a}{\varphi_1+\varphi_2}, \qquad a=1,2 \tag{2.5}$$

$$g^{ai}=0, \qquad a\neq i \tag{2.6}$$

$$g^{\alpha\beta}=\frac{1}{\varphi_1+\varphi_2}\left(\zeta_1^{\alpha\beta}\psi_1+\zeta_2^{\alpha\beta}\psi_2\right), \qquad \alpha,\beta=3,4 \tag{2.7}$$

where φ_a, ψ_a, and $\zeta_a^{\alpha\beta}$ are functions of the (nonignorable) coordinate x^a only. The nontrivial K tensor K_{ij} is then given by

$$K^{11}=\frac{\varphi_2\psi_1}{\varphi_1+\varphi_2}, \qquad K^{22}=-\frac{\varphi_1\psi_2}{\varphi_1+\varphi_2} \tag{2.8}$$

$$K^{ai}=0, \qquad a\neq i \tag{2.9}$$

$$K^{\alpha\beta}=\frac{1}{\varphi_1+\varphi_2}\left(\zeta_1^{\alpha\beta}\psi_1\varphi_2-\zeta_2^{\alpha\beta}\psi_2\varphi_1\right), \qquad \alpha,\beta=3,4 \tag{2.10}$$

We recall that a K tensor K_{ij} has *Segrè characteristic* [(11)(11)] if it admits two double eigenvalues, say A and B. In Benenti and Francaviglia

(1980), Section 6, it was pointed out that the K tensor (2.8)–(2.10) which characterizes a $\mathbb{S}_2$-separability structure in a space-time (V_4, g) has Segrè characteristic [(11)(11)] if and only if the following condition holds:

$$\det \| \zeta_a^{\alpha\beta} \| = 0, \qquad a = 1,2 \tag{2.11}$$

Condition (2.11) is one of Carter's hypotheses (see Carter, 1969): it actually characterizes Carter's separable space-times among all $\mathbb{S}_2$-separable space-times (see Benenti and Francaviglia, 1980, Section 6; and Francaviglia and Virga, 1980).

It is also clear the in a vacuum type-D space-time a nontrivial K tensor with Segrè characteristic [(11)(11)] can be written as follows:

$$K_{ij} = A(l_i n_j + n_i l_j) + B(m_i \bar{m}_j + \bar{m}_i m_j) \tag{2.12}$$

where $(l, n, m, \bar{m})$ is a null tetrad associated with the GSF congruences.

3. KILLING–YANO TENSORS IN VACUUM TYPE-*D* SPACE-TIMES

We recall that a Killing–Yano tensor[6] is a skew symmetric 2-tensor f_{ij} such that

$$\nabla_m f_{ij} + \nabla_j f_{im} = 0 \tag{3.1}$$

If f_{ij} is a KY tensor, its square

$$K_{ij}(f) = f_{im} f_j^{\ m} \tag{3.2}$$

is a K tensor. KY tensors in space-time have been investigated by Collinson, who proved the following results:

(i) An irreducible K tensor K_{ij} is the square $K(f)$ of a KY tensor f_{ij} only if K_{ij} has Segrè characteristic [(11)(11)] (see Collinson, 1976, Theorem 1).

(ii) Let us have a K tensor K_{ij} of the form (2.12). Let us take

$$\tilde{f}_{ij} = A^{1/2}(l_i n_j - l_j n_i) + B^{1/2}(m_i \bar{m}_j - \bar{m}_i m_j) \tag{3.3}$$

so that

$$K_{ij} = K(\tilde{f})_{ij} \equiv \tilde{f}_{im} \tilde{f}^{m}{}_{j} \tag{3.4}$$

[6]Hereafter abbreviated KY tensor.

Then $\tilde{f}$ is a KY tensor provided $a = A^{1/2}$ and $b = B^{1/2}$ satisfy the following conditions (in Newman–Penrose formalism):

$$\begin{aligned} a(\tau+\bar{\pi}) &= ib(\tau-\bar{\pi}) \\ a(\rho+\bar{\rho}) &= ib(\rho-\bar{\rho}) \\ a(\mu+\bar{\mu}) &= ib(\mu-\bar{\mu}) \end{aligned} \tag{3.5}$$

By applying the above to vacuum type-D space-times we can easily find among them all those solutions which admit a KY tensor. This analysis was first carried out by Collinson (1976) and recently by Stephani (1978). However, in Collinson (1976) it was erroneously claimed that also the C solution belongs to the family. The mistake arose from the supposition that conditions (3.5) are satisfied by Robinson–Trautman space-times. By using the table for type-D space-times given in Demianski and Plebanski (1976) to check conditions (3.5) we easily realize that *all vacuum type-D solutions without acceleration admit a KY tensor*. This KY tensor is, of course, the one already computed by Collinson: its square provides us with a K tensor of Segrè characteristic [(11)(11)], hence giving another proof of the existence of a K tensor in all those space-times (see Demianski and Francaviglia, 1980, Section 3).

4. CONCLUSIONS

To summarize, we can formulate the following theorem.

Theorem. Let (V_4, g) be a vacuum type-D space-time. The following conditions are equivalent:

(i) (V_4, g) is without acceleration.

(ii) (V_4, g) admits a $\mathfrak{S}_2$ separability structure and the metric g in canonical form satisfies condition (2.12).

(iii) (V_4, g) is one of Carter's separable space-times.

(iv) (V_4, g) admits a K tensor of Segrè characteristic [(11)(11)].

(v) (V_4, g) admits a KY tensor.

The equivalence of the above statements easily follows from the remarks of Sections 2 and 3. The theorem provides a link between several isolated pieces of information and it clarifies the relationship between separability and the existence of Killing and Killing–Yano tensors.

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